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Simple equation for earthquake distribution

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The earthquake distribution, the Gutenberg-Richter law, is obtained postulating a connection through a diffusion process between the kinetic energy of the active zone and the radiated energy. A link between the value of the critical exponent and the stress release is proposed. [S1063-651X(99)01306-9]

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Earthquakes usually appear at the contact zone of the tectonic plates, which form the crust of the Earth [1]. The generic mechanism consists in the breaking along a fault of deformed rocks, but the detailed features of this phenomena has not been completely understood yet [2,3]. The scaling laws describing the large-scale properties of the earthquakes [4,5] rise the question whether the ideas of the nonequilibrium critical phenomena theory may be helpful in further understanding this phenomenon.

A fundamental observation in seismology is the Gutenberg-Richter law [4]; from it one can infer that the probability of the earthquakes decays algebraic function of the released energy,

$$P(E)\approx E^{-1-b}. \tag{1}$$

The actual value of the exponent b is still under discussion; earlier it was reported to be close to 1 but at the moment it is believed to be about 2/3. It has been observed that b varies for different Earth regions, fluctuates in time, and depends on the earthquake magnitude [6–8]. In the recent years self-organized-criticality models have been proposed to explain this scaling law [5,6,9–11] as the result of the extremal nature of the dynamic rules governing the system [12]. The results of these models were mainly obtained through numerical simulation. There are also deterministic models [2,8] describing the earthquakes dynamics starting from a more realistic description of the friction and elastic forces acting in the fault zone. The Budridge-Knopoff kind of model [13] reproduces the Gutenberg-Richter law with $b\approx 1$.

In this paper we propose an analytic model based on a birth and death process whose parameters are deduced from the mean field type of arguments. The model explains the scale invariance observed in the Gutenberg-Richter law and also accounts for the observed variability of the critical exponent b. The use of birth and death processes and more complex multiplicative random process have been used previously to model scale-invariant physical phenomena [14].

The stress accumulated in the fault zone appears due to the very slow motion of the tectonic plates driven by the underlying magma. As the motion is very slow we assume that the thermodynamic equilibrium is satisfied locally in the fault zone and its motion can be depicted in principle using a hydrodynamic approach. This picture breaks down when the friction forces cannot equilibrate in some regions the stress forces accumulated in the fault zone; consequently, the velocity and the stress fields became discontinuous in these regions and the induced dynamics violates the local thermodynamic equilibrium condition. These are the regions that produce the seismic waves and we shall call them active zones. The initial-produced perturbation induces elastic failure in other regions that are close to the hydrodynamic break-down point.

In this process the dynamics of the active zones is deterministic but chaotic as models show [13]; therefore, we assume that the kinetic energy that characterizes the active zones after the elastic failure is evolving randomly due to complexity of the elastic energy landscape of the fault zone. If an active part has an average kinetic energy of order BL^2 where B is bulk modulus and L is its linear dimension we assume that there is a characteristic length of order $\sqrt{E_{kin}/B}$ above which the correlation between two active points vanish.

We model this picture denoting the independent active volumes as active points carrying an average kinetic energy that is exchanged among them and that is lost finally from the fault zone by radiation. The active point correspond to the ''blocks'' or lattice sites of the above-cited models. Next, we propose a simple stochastic rule for the evolution of the active points in the hypotheses that they are statistically independent.

We consider that during the release of a small quantity of energy dE outside of the fault zone the number of active points varies randomly. In this simple model we assume that an active point has the probability λdE to die and the probability μdE to generate another active point. This is a very schematic description of the kinetic-energy evolution in the fault zone. These two parameters are the average effects of the interaction of the active points among them and with the surrounding media. With the above hypothesis we can write a detailed balance equation for the numbers of active agents using as evolution parameter the released energy,

$$P_n(E+dE)=P_n(E)+\mu(n-1)dEP_{n-1}(E)+\lambda(n+1)dEP_{n+1}(E)-(\lambda+\mu)ndEP_n(E), \tag{2}$$

where $P_n(E)$ is the probability to have n active points after the energy E has been released; μndE, λndE are the probabilities that the system jumps from the state with n active points in a state with $n+1$, $n-1$ active points while it releases the energy dE. Since we consider the active points to be statistically independent the transitions with larger steps are proportional with higher powers of dE and, therefore, we neglect them.

If we neglect the interaction of the active points with the surrounding media and the friction among them the birth and death rates are equal; $\lambda=\mu$ on the basis of the energy conservation on average. If an active point appears, carrying a typical amount of kinetic energy, an other agent has to disappear. (Later on we shall discuss the situation in which the death rate incorporates the effect of internal friction such that $\lambda>\mu$.) To this first approximation we have to add two more effects on the transition probabilities: the loss of energy by seismic waves and the input of kinetic energy coming from the stress release of the fault zone during the earthquake. This correction may be set to the death rate λ and it is inverse proportional with the number of active point,

$$\lambda\rightarrow\lambda+u/n, \tag{3}$$

considering that the loss or gain of energy is equally distributed among the active points. In the previous formula u is the global rate of energy transfer between the active points and the fault. If the rate of the stress release dominates the rate of radiated energy $u<0$, otherwise $u>0$. Intuitively, we expect that the stress release will dominate at the beginning of the earthquake but now we are considering an average value of this effect.

Within the frame of this model an earthquake begins with a given number of active points n_0 and comes to end when the number of active points, and implicitly the kinetic energy is zero. Therefore, the earthquake distribution is the distribution of the first arrival in the origin of the random walk described by Eq. (2). The calculation of this distribution can be done in the continuous version of the model,

$$\partial_E p_E(n)=\tfrac{1}{2}\partial_{nn}(Dnp_E(n))+v\,\partial_n p_E(n), \tag{4}$$

where D and v are the phenomenologic parameters corresponding in the discreet version to λ and u. D is the variance of the average kinetic-energy per active point and v is the drift added due to kinetic-energy variation coming to the interaction with the surrounding media.

The solution of Eq. (4) can be obtained using the time Laplace transform and then the standard theory of the ordinary differential equation [15,16]. The Laplace transform of the adjoint (backward) equation is

$$np''-rp'-\lambda p=p(n,0), \tag{5}$$

where we made the substitutions: $n\rightarrow Dn$, $r=v/D$. The Laplace transform of the first arrival at n_0 starting from n, $\phi_\lambda(n,n_0)$ satisfies the homogeneous equation associated with Eq. (5) [15] and it has to be a bounded function as $n\rightarrow\infty$. The homogeneous equation associated with Eq. (5) can be solved exactly and the Laplace transform of the first arrival distribution has the form [15],

$$\phi_\lambda(n,n_0)=\left(\frac{n}{n_0}\right)^{(1-r)/2}\frac{K_\nu(2\sqrt{\lambda}n^{1/2})}{K_\nu(2\sqrt{\lambda}n_0{}^{1/2})}=\left(\frac{n}{n_0}\right)^{(1-r)/2}\frac{I_{-\nu}(2\sqrt{\lambda}n^{1/2})-I_\nu(2\sqrt{\lambda}n^{1/2})}{I_{-\nu}(\sqrt{2\lambda}n_0{}^{1/2})-I_\nu(2\sqrt{\lambda}n_0{}^{1/2})}, \tag{6}$$

where $K_\nu(z)$ and $I_\nu(z)$ are the Bessel function of imaginary argument of first and second kind, respectively [17], and $\nu=1-r$. Using the series representation of the Bessel function $I_\nu(z)$ one can analyze the behavior of the function $\phi_\lambda(n,n_0)$ as $\lambda\rightarrow0$. That is equivalent through the Tauberian theorem with the limit $E\rightarrow\infty$. We conclude [18,15] that the asymptotic behavior of the probability distribution for the earthquake distribution function of released energy is

$$p(n_0,E)\approx E^{-1-b},\quad b=1+r,\quad r=v/D. \tag{7}$$

The above equation yields the Gutenberg-Richter law if the drift part is less than the noise strength, that is, $|r|<1$; this request is compatible with the picture we have proposed since during the earthquake event, as we assumed the breakdown of the hydrodynamic description, the variation of the kinetic energy of the active points is drifted by the noise strength D. The observed variability of the exponent b originates from the global characterization of the interaction between the active part and its surrounding area through parameter r that may vary for various places on the Earth and for different events as the configuration and the thermodynamic properties may change during the hydrodynamic regime.

In the frame of this model one can explain the relation $b<1$. This implies $u<0$ in Eq. (3), that is, the stress release rate is greater than the kinetic energy rate loss by seismic waves.

The asymptotic behavior we have found remain unchanged upon the average over the distribution of the initial activity if we assume that this distribution has all its moments finite. From a physical point of view, it is natural to assume that the earthquake starts with a number of initial active points and whose distribution decays exponentially with n_0.

Now we consider briefly the case when the dissipation is present in the volume of the active zone, that is, the death rate λ is higher than the birth rate μ. The drift term in the diffusion equation becomes $v-\epsilon n$, $\epsilon>0$. The equation for the first arrival time is

$$np''-(\epsilon n-r)p'-\lambda p=0. \tag{8}$$

The substitution $x \rightarrow n/\epsilon$ lead us to the equation satisfied by confluent (degenerate) hypergeometric function,

$$xp'' + (r-x)p' - \frac{\lambda}{\epsilon}p = 0. \quad (9)$$

Standard calculation shows that the Laplace transform of the first arrival distribution is

$$p_{\lambda}^{fa}(n_0) = \Psi\left(\frac{\lambda}{\epsilon}, r, \epsilon n_0\right), \quad (10)$$

where $\Psi(a,b;z)$ is the hypergeometric confluent function bounded to $+\infty$ [17,16]. The Laplace transform of the first arrival in the origin is now analytical in λ and the cutoff energy goes proportionally with $1/\epsilon$. Therefore, the system self-organizes in a critical state if the internal friction in the active zone is negligibly small comparatively with the kinetic energy released after the elastodynamic failure; otherwise, the large-scale events are prohibited. A low friction motion appears if the energy flux dissipated by internal friction is much more less than the energy released through the radiative modes. This kind of dynamics can be imagined if we assume that the fault zone evolves in a rugged landscape of energy; an earthquake implies an abrupt transition between two local minima and the short characteristic time of this transition activates the radiative modes of the system. A model sustaining this hypothesis has been treated in Ref. [2].

We have shown that the Gutenberg-Richter law can be obtained analytically from a model using as evolution parameter, the released energy E. The parameter b includes the average effect of the interaction between the active zone and surrounding media which can explain the variability of the critical exponents observed in nature. The relation $b<1$ is connected with the relaxation in the fault zone. The Markovian hypothesis made by this paper may be useful in the further study of the simple deterministic model [2,8] since one clearly can individuate the active points from the active zone and the radiated energy and check their stochastic correlation.

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